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# Toward a turbulent constitutive relation

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In an attempt to explain the failure of the various pure homogeneous strain experiments to reach equilibrium (and consequently to support the contention of Townsend of an equilibrium structure of the Reynolds stress dependent only on geometry), the nature of the general Reynolds stress-mean velocity relation is examined. It is shown that if homogeneous flows become asymptotically independent of initial conditions, and if the Reynolds stress bearing structure can be characterized by a single time scale (i.e.—at sufficiently high Reynolds number) then these flows behave like classical non-linear viscoelastic media, with the Reynolds stress structure dependent on the (strain-rate) (time scale) product. Thus, the existence of an equilibrium structure implies the existence of an equilibrium time scale and a universal value of the product. The ideas permitting Reynolds stress and mean velocity to be related are applied to the dissipative structure in homogeneous flows, and it is found that in such flow the time scale never ceases to grow, so that these flows can never reach an equilibrium structure. With the aid of an *ad-hoc* assumption these flows are examined in some detail, and the results of experiments are predicted with considerable accuracy. It is suggested that (inhomogeneous) flows having an equilibrium time scale may, in the homogeneous limit, be expected to display a universal structure. The small departure from universality induced by the large eddies associated with inhomogeneity may be adequately predicted by this same *ad-hoc* model.

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## CONTENTS



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## 1. Introduction

In 1956 Townsend suggested that turbulence in energetic equilibrium had a structure dependent only on the geometry of the (homogeneous, steady) strain rate field. A plane homogeneous pure strain experiment (Townsend 1954) appeared to support the contention of an equilibrium structure, although the apparent approach to equilibrium was so close to the end of the duct as to cause suspicion. In 1967 Maréchal repeated Townsend's (1954) experiment, but in an extended duct, and in 1968 Tucker and Reynolds did the same independently. Both obtained a higher degree of anisotropy than Townsend, with the same suspicious levelling-off just before the end of the duct. These experiments placed in question the concept of an asymptotic structure dependent only on geometry, although this idea appears to be consistent with observations in inhomogeneous shear flows.

We will attempt to answer the question of whether homogeneous turbulence undergoing deformation can ever be expected to attain an equilibrium structure. We will approach this question by asking first (in appendix A) whether and how the Reynolds stress and other statistical functions of the fluctuating velocity field can be considered to be functionals of the mean velocity field. Using the results of appendix A, for turbulence characterized by a single time scale, the Reynolds stress is seen to be self-preserving. The answer to our question regarding an equilibrium structure is thus seen to depend on the evolution of the time scale: if and only if the time scale reaches an equilibrium value, the structure will reach equilibrium. The results of appendix A are now applied to the dissipative structures, leading to the result that the time scale grows continually, so that no equilibrium structure is achieved. Finally, a model is constructed on the basis of appendix A (very similar to one proposed on somewhat different grounds by Rotta (1951)), which is used to predict with considerable qualitative success the results of the three homogeneous pure strain experiments.

As a bonus, the considerations of appendix A indicate that turbulence undergoing homogeneous deformation behaves like a classical non-linear non-Newtonian medium, and in particular the inequality of the normal components of the Reynolds stress in the plane of deformation is seen to be direct evidence of viscoelastic behaviour. This has frequently been hinted at (see Rivlin (1957) and Liepmann (1961), who both suggested that the secondary motions or large eddy structure of turbulent shear flows could be explained by the validity of some form of non-Newtonian non-linear constitutive relation), but previously was supported only by linearized calculations for short times or small strains (see, for example, Moffatt (1965) and Crow (1968)) which displayed linear elastic response, and by a demonstration that such response could qualitatively explain phenomena observed at the turbulent–non-turbulent interface (Townsend 1966).

## 2. Self preservation

From appendix A we find that in either homogeneous shear or pure homogeneous strain the Reynolds stress can be written in terms of invariant functions (respectively four and three) of the strain rate (in the former case because the vorticity is proportional to the strain rate). As discussed in §A 3, this must be non-dimensionalized by a time scale provided by initial conditions (since there are no boundary conditions in a homogeneous flow); the invariant functions will in addition be functions of the ratios of the various time and length scales determined by the initial conditions, if there are others, as well as the Reynolds number.

Let us consider the case of the evolution of turbulence characterized by a single time and length scale, not an unreasonable possibility sufficiently long after initiation. To achieve this we must assume the Reynolds number to be sufficiently large so that it has no direct influence on the Reynolds stress. We will then have something like

$$R_{ij} = q^2 f_{ij}(U'q^2/\epsilon). \tag{2.1}$$

From §A 3 we might at first expect (2.1) to be a functional of the development of $q^2/2\epsilon$; if only a single scale is relevant, however, this development is governed also by $q^2/2\epsilon$, so that given the present value, the development is universal.

From (2.1) it is evident that Townsend's conjecture of an equilibrium structure is completely dependent on the time scale attaining an equilibrium (and universal) value. This makes physical sense: one would expect in a flow in equilibrium that all time scales would be proportional. Hence, we must examine the evolution of the time scale.

## 3. Evolution of the time scale

Consider a homogeneous turbulent flow with homogeneous deformation. Then, writing $\epsilon = \nu\overline{u_{i,k}u_{i,k}}$, manipulations with the equations of motion yield

$$\frac{\dot{\epsilon}}{2\nu} + U_{i,j}\overline{u_{i,k}u_{j,k}} + U_{j,k}\overline{u_{i,k}u_{i,j}} + \overline{u_{i,k}u_{i,j}u_{j,k}} = -\nu\overline{u_{i,kj}u_{i,kj}}. \tag{3.1}$$

Now, we may apply the reasoning of appendix A to the various terms in (3.1); for example,

$$\overline{u_{i,k}u_{j,k}} = \mathscr{F}_{ij}\{\mathsf{C}(t/t'),\quad \mathbf{\Omega}(t/t'),\quad R_{L\epsilon}\},\quad t' \leqslant t, \tag{3.2}$$

where no spatial dependence has been indicated, since the deformation is homogeneous. We expect a time scale characteristic of the dissipative range of wave-numbers to be $\surd(\nu/\epsilon)$, and we expect (3.2) also to be a function of Reynolds number $R_{L\epsilon}$. Whereas the time scale of the Reynolds stress may always be expected to be of the order of the inverse of the mean strain rate, this is not true of the time scale of the dissipation. The ratio of the two time scales, in fact, is of order of $R_{L\epsilon}^{\frac{1}{2}}$, so that we expect the time scale $\surd(\nu/\epsilon)$ to become shorter and shorter relative to that of the mean motion as the Reynolds number increases. Thus, we have some justification for carrying out an expansion similar to that used by Coleman & Noll (1961).† This may be done in many ways, but perhaps the easiest is to introduce a 'retarded history' by means of a coefficient $0 \leqslant \alpha \leqslant 1$;

$$\overline{u_{i,k}u_{j,k}} = \mathscr{F}_{ij}\{\mathsf{C}(t/t-\alpha(t-t')), \quad \mathbf{\Omega}(t/t-\alpha(t-t')), \quad R_{L\epsilon}\}, \quad t-t' \geqslant 0 \tag{3.3}$$

and we may expand in a series in $\alpha$, keeping only the leading terms (implying that the $S\surd(\nu/\epsilon) \ll 1$). Thus

$$\overline{u_{i,k}u_{j,k}} = \frac{\epsilon}{3\nu}(\delta_{ij}+\beta\surd(\nu/\epsilon)S_{ij}+\ldots). \tag{3.4}$$

In an exactly similar way we can write

$$\left.\begin{aligned}\overline{u_{i,kj}u_{i,kj}} &= \frac{\epsilon^{\frac{3}{2}}}{\nu^{\frac{3}{2}}}(A+O(S^2\nu/\epsilon)),\\ \overline{u_{i,k}u_{i,j}u_{j,k}} &= \frac{\epsilon^{\frac{3}{2}}}{\nu^{\frac{3}{2}}}(B+O(S^2\nu/\epsilon)),\end{aligned}\right\} \tag{3.5}$$

where $O(S^2)$ is taken to include not only $\mathrm{tr}(\partial\mathsf{C}/\partial t')^2_{t'=t}$, but also $\mathrm{tr}(\partial^2\mathsf{C}/\partial t'^2)|_{t'=t}$, $(\partial\mathbf{\Omega}/\partial t')\cdot(\partial\mathbf{\Omega}/\partial t')|_{t'=t}$, etc.

The quantities $A$, $B$ are taken to be functions of

$$R_{L\epsilon} = \frac{qL_\epsilon}{3^{\frac{1}{2}}\nu}, \quad L_\epsilon = q^3/3^{\frac{3}{2}}\epsilon.$$

Substituting the expressions (3.4) into (3.1) we can write

$$\dot{\epsilon} = -2(A+B)\epsilon^{\frac{3}{2}}/\nu^{\frac{1}{2}} + O(S^2\nu^{\frac{1}{2}}\epsilon^{\frac{1}{2}}). \tag{3.6}$$

The remainder is of order (presuming an equilibrium value estimated from Rose (1966) $Sq^2/\epsilon \sim \surd 10$)

$$\frac{S^2\nu}{\epsilon(A+B)} = O\left(\frac{1}{R_{L\epsilon}(A+B)}\right). \tag{3.7}$$

Presuming that $R_{L\epsilon}(A+B)\to\infty$, $R_{L\epsilon}\to\infty$, we may neglect the remainder. In general $R_{L\epsilon}$ will be a function of time; this is not inconsistent with the forms chosen in (3.4) and (3.5), since these are expansions about a state of instantaneous response to changing conditions—infinitesimal time constant. Hence, $(A+B)$ will in general be a function of time. In grid turbulence, however, which is one of the flows described by (3.6), $R_{L\epsilon}$ is constant. Let us solve (3.6) (neglecting the remainder) for $(A+B)$ constant. We obtain

$$\epsilon = \nu/(A+B)^2(t-t_0)^2, \tag{3.8}$$

† In Lumley (1967) it was shown by a similar series expansion that the correction for inhomogeneity is of higher order than that for time dependence. There, however, the dependence on $\mathbf{\Omega}$ was neglected. If the dependence on $\mathbf{\Omega}$ is included, it is not difficult to show that the first-order time term is unchanged, the second-order time term is increased by a term quadratic in the vorticity, and a term proportional to the vorticity gradient appears, bi-linear in space and time.

where $\epsilon \to \infty$ at $t_0$. Since in this flow $-\dot{q}^2 = 2\epsilon$, we obtain

$$q^2 = 2\nu/(A+B)^2(t-t_0), \tag{3.9}$$

so that

$$R_{L\epsilon} = \frac{4}{9(A+B)^2} \tag{3.10}$$

or

$$A+B = \tfrac{2}{3}R_{L\epsilon}^{-\frac{1}{2}}. \tag{3.11}$$

Since the forms (3.5) must be independent of the presence or absence of mean shear, (3.11) holds also where $R_{L\epsilon}$ is not constant. Now we may evaluate the remainder term in (3.6) as $O(R_{L\epsilon}^{-\frac{1}{2}})$, so that our neglect was justified. The conclusion (3.11) corresponds to the concept of an 'equilibrium range' (Batchelor 1956); that is, we anticipate that $(\dot{\epsilon}/\epsilon)\surd(\nu/\epsilon) \to 0$ as $R_{L\epsilon} \to \infty$, so that the rate of change of the dissipating region is small, measured in its own time scales.

Since $R_{L\epsilon}$ has always the same sign, we may conclude that in a homogeneous turbulence $\epsilon$ will continue to decay whether there is a mean strain rate or not, if the Reynolds number is high enough; evidently $\epsilon$ is fixed in practical flows by spatial transport induced by inhomogeneity. Using (3.11) we may rewrite (3.6) as

$$\dot{\epsilon}/\epsilon = -4\epsilon/q^2, \tag{3.12}$$

the same form as in grid turbulence. The rate of change of the time scale $q^2/2\epsilon$ may be calculated as (setting $T = q^2/2\epsilon$)

$$\dot{T}/T = \dot{q}^2/q^2 - \dot{\epsilon}/\epsilon = \dot{q}^2/q^2 + 4\epsilon/q^2. \tag{3.13}$$

From the equations of motion, we have

$$\dot{q}^2 + 2U_{i,j}R_{ij} = -2\epsilon, \tag{3.14}$$

so that

$$\dot{T}/T = 2(\epsilon - S_{ij}R_{ij})/q^2. \tag{3.15}$$

Since $S_{ij}R_{ij} \leqslant 0$ virtually always, and $\epsilon \geqslant 0$, the time scale continually increases; the greater the production, the greater the rate of increase. Since $T$ is monotone increasing in $t$, any function of $t$ may be expressed as a function of $T$, justifying the concept embodied in (2.1).

Consequently, we do not expect homogeneous flows to reach an equilibrium structure, but to evolve monotonously as the time scale increases.

## 4. A model for the development of Reynolds stress in homogeneous flows

We now have a qualitative conclusion, asymptotically valid at large Reynolds number: that turbulence under homogeneous deformation behaves like a classical non-linear viscoelastic fluid, but having a monotone increasing time scale, so that no equilibrium structure can be achieved. This conclusion is relatively assumption free; we have used only the assumption that detailed initial conditions will be forgotten. Aside from that, we have used only group-theoretical arguments. To obtain quantitative results for comparison with experiment, we must (since we do not have a mechanism for solving the equations exactly) make a simplifying assumption consistent with the qualitative behaviour that we have found.

Let us begin in the usual way, by writing the equations for the Reynolds stress tensor in a homogeneous deformation as (with $\overline{u_i u_j} = R_{ij}$)

$$\dot{R}_{ik} + U_{i,j}R_{jk} + U_{k,j}R_{ji} = -(1/\rho)(\overline{u_k p_{,i}} + \overline{u_i p_{,k}}) - 2\nu\overline{u_{i,j}u_{k,j}}. \tag{4.1}$$

Both terms on the right-hand side present difficulties. Let us dispose of the second one first. From (3.4) we can write

$$\nu\overline{u_{i,j}u_{k,j}} = \epsilon\delta_{ik}/3 + \ldots, \tag{4.2}$$

where the higher order terms are of order $v^2S$, where $v$ is the Kolmogorov velocity, and $S$ is the strain rate. For high enough Reynolds number, then, these may be neglected relative to the second and third terms on the left-hand side.

The first term on the right-hand side presents greater difficulties. It represents transfer of energy among the components. On a somewhat *ad-hoc* basis,† we propose the following

$$-1/\rho(\overline{u_i p_{,k}} + \overline{u_k p_{,i}}) = (1/T)(q^2\delta_{ik}/3 - R_{ik}). \tag{4.3}$$

Equation (4.3) was first proposed by Rotta (1951) for a portion of the Reynolds stress (that part dependent on triple correlations; the other part was expressed as a series in successive derivatives of the mean velocity profile). We propose (4.3) on the following grounds: through equation (4.1) and (4.2), the symmetric part of the pressure gradient-velocity correlation could be written as a functional of the history of the Reynolds stress (the equations of mean motion being used to replace the mean velocity gradient by dependence on the Reynolds stress), with all the reservations regarding initial and boundary conditions that are expressed in appendix A. If now we imagine that the time scale characteristic of the pressure gradient-velocity correlation is short relative to that for the change of the Reynolds stress (necessarily in intrinsic co-ordinates, and hence the same as that of the strain rate), then we may expand; the leading term depends only on the present value of the Reynolds stress, and hence must have the same eigenvectors. The pressure gradient-velocity correlation has only two independent eigenvalues in a three-dimensional homogeneous flow; in a two-dimensional one, only one. In a two-dimensional flow, then, (4.3) would be exact (granted the short time scale expansion). In a quasi-two-dimensional flow it may not be too bad.

We are assuming, of course, that $T > 0$, since if $R_{33} < q^2/3$, we want $-2/\rho\overline{u_3 p_{,3}} > 0$; $T$ is the time constant associated with the return to isotropy. Equation (4.3) can be read as stating that the rate at which energy is fed to a

† For experimental support, see the paper by Corrsin in this Symposium. This assumption is not inconsistent with Crow (1968) where it is shown for weak strains that part of the pressure field arises from interaction of the mean field with the turbulence. To the order of Crow's analysis, both sides of (4.3) will have the eigenvectors of the strain rate field. A developed turbulence will acquire an anisotropy determined by (but not necessarily the same as that of ) the mean field, so that either one may be taken as argument in expressing the anisotropy of another quantity in the same flow.

component depends on the energy deficit in that component below the mean energy level. This is a sort of *stosszahlansatz*, like that of classical statistical mechanics. Since it specifies rates dependent on present conditions, and hence implies Markovian behaviour, it is surely wrong (a reflexion of the fact that our time scale expansion cannot possibly be justified). The fact that it produces equations very similar to those governing the behaviour of macromolecules in dilute solution is evidence of its family relationship to the assumptions of statistical mechanics. It may be regarded as the equivalent for the pressure-velocity correlation of the eddy viscosity approximation for the Reynolds stress.

We would expect to meet with difficulty in applying (4.3) in a homogeneous shear since the principal axes would not be expected to be the same. In a pure homogeneous strain, the axes must be the same, so we would expect the approximation to be better; difficulties, if any, would arise only from the assumption of linearity.

We may examine the behaviour of (4.3) in the time-dependent case with no mean flow:

$$\dot{R}_{ik}+(1/T)(R_{ik}-q^2\delta_{ik}/3) = -2\epsilon\delta_{ik}/3. \tag{4.4}$$

Forming the equation for $q^2$, $\dot{q}^2 = -2\epsilon$, and subtracting, we have

$$\partial(R_{ik}-q^2\delta_{ik}/3)/\partial t+(1/T)(R_{ik}-q^2\delta_{ik}/3) = 0. \tag{4.5}$$

Comparing the time scale for return to isotropy with that for decay, we have

$$\begin{aligned}(1/q^2)(dq^2/dt) &= -2\epsilon/q^2,\\ (R_{ik}-q^2\delta_{ik}/3)^{-1}\,d(R_{ik}-q^2\delta_{ik}/3)/dt &= -1/T\end{aligned} \tag{4.6}$$

(no sum on indices). We know that the return to isotropy is faster than the decay —hence we must have

$$1/T > 2\epsilon/q^2. \tag{4.7}$$

Let us set $2\epsilon T/q^2 = \beta \leqslant 1$, where $\beta$ is a constant. Using (3.12) for the evolution of $\epsilon$, we have

$$\dot{\epsilon}/\epsilon = -4\epsilon/q^2, \tag{3.12}$$

from which we obtain in this shear free case, $T = T_0+\beta t$. Substituting in (4.6) and solving, we obtain

$$R_{ik}-q^2\delta_{ik}/3 = [R_{ik}-q^2\delta_{ik}/3]_0(1+\beta t/T_0)^{-1/\beta}, \tag{4.8}$$

where $[\quad]_0$ quantities are evaluated at $t = 0$. This behaviour, in which the time constant for the return to isotropy becomes progressively longer as time goes on, is qualitatively consistent with observation. Virtually the only data on the return to isotropy is contained in the experiment of Tucker & Reynolds (1968) (which displayed an encouraging return to isotropy, contrary to that of Grant (1958)). There the anisotropy is presented in terms of $K = (R_{22}-R_{11})/(R_{22}+R_{11})$. According to (4.8) and (4.6) we obtain

$$K = K_0(1+\beta t/T_0)^{1-1/\beta}/\{2q_0^2/3(R_{11}+R_{22})_0+[1-2q_0^2/3(R_{11}+R_{22})_0] \times(1+\beta t/T_0)^{1-1/\beta}\}. \tag{4.9}$$

For initial conditions we have, estimating from Tucker & Reynolds graphs,

$$\left.\begin{aligned} 2q_0^2/3(R_{11}+R_{22})_0 = 1{\cdot}05, \quad K_0 = 0{\cdot}62, \quad dK_0/dt &= -2{\cdot}46\ \text{sec}^{-1}, \\ T_0 &= 0{\cdot}828\,\beta\ \text{sec}. \end{aligned}\right\} \tag{4.10}$$

The figure for $T_0$ is very difficult to determine directly from the decay data; the value given was estimated from (3.12) by integrating the values for $q^2$ given by Tucker & Reynolds. Using the values (4.10), we obtain for $\beta$

$$\beta = 0{\cdot}242. \tag{4.11}$$

If we now use these same values to obtain the value 0·292 seconds later, read from Tucker & Reynolds graphs as 0·32, we obtain 0·234, which is too low. We must conclude that either the approximation of $\beta =$ const. (essentially a similarity assumption, that all time scales are proportional) is not valid for substantial changes in time scale, or that the assumption of linearity in (4.3) is poor.

## 5. Homogeneous shear

Returning now to the more general case, let us consider the equations (4.1) for a steady homogeneous shear. Introducing a new time variable defined by

$$d\tau/dt = 2\epsilon/\beta q^2, \tag{5.1}$$

we find immediately from (3.12) that $\epsilon = \epsilon_0 e^{-2\beta\tau}$, and the equations (4.1) become (where $(\ )' = \partial(\ )/\partial\tau$, we have presumed $R_{13} = R_{23} = 0$ initially, and we have used (4.3)):

$$\left.\begin{aligned} R'_{11}+R_{11} &= U'\beta q^2 R_{12}/\epsilon + (1-\beta)q^2/3, \\ R'_{22}+R_{22} &= (1-\beta)q^2/3, \\ R'_{33}+R_{33} &= (1-\beta)q^2/3, \\ R'_{12}+R_{12} &= -U'\beta q^2 R_{22}/2\epsilon. \end{aligned}\right\} \tag{5.2}$$

The exact solution of these may be written down, but there is hardly any point in it, since the solution depends implictly on $q^2$; an equation for $q^2$ may be obtained by adding together those for $R_{11}$, $R_{22}$ and $R_{33}$, but it cannot be solved explicitly. It is more convenient to define

$$\theta = U'\beta q^2/2\epsilon, \quad \mathscr{R}_{ij} = R_{ij}/q^2 \tag{5.3}$$

and rewrite (5.2) as

$$\left.\begin{aligned} d\mathscr{R}_{11}/d\theta &= [2\theta\mathscr{R}_{12}(\mathscr{R}_{11}-1)+(\beta-1)(\mathscr{R}_{11}-\tfrac{1}{3})]/[\beta\theta-2\theta^2\mathscr{R}_{12}], \\ d\mathscr{R}_{22}/d\theta &= [2\theta\mathscr{R}_{22}\mathscr{R}_{12}+(\beta-1)(\mathscr{R}_{22}-\tfrac{1}{3})]/[\beta\theta-2\theta^2\mathscr{R}_{12}], \\ d\mathscr{R}_{12}/d\theta &= [-\theta\mathscr{R}_{22}+\mathscr{R}_{12}(\beta-1+2\theta\mathscr{R}_{12})]/[\beta\theta-2\theta^2\mathscr{R}_{12}]. \end{aligned}\right\} \tag{5.4}$$

We are particularly interested in comparing our predicted values with those of Rose (1966), an experimental realization of this flow. To this end, equations (5.4) were solved by the Runge–Kutta technique on the IBM 360/67, with initial conditions $R_{11} = R_{22} = R_{33} = \frac{1}{3}$, $R_{12} = 0$ at $\theta = 0{\cdot}24$, using the value $\beta = 0{\cdot}242$. The results are shown in figure 1, together with the results of Rose, read from his faired curves and calculated to place them in this form. The values of $\epsilon$ needed to calculate $\theta$ were obtained by integrating equations (3·12), using Rose's values of $q^2$.

The general agreement is excellent. The value of $\theta$ obtained from equation (3.12) agrees, near the end of the duct, with the value of $\theta$ computed there (at an extremum of $q^2$) on the basis of equilibrium. Plotted in this form, a distinct change in curvature is evident in all the curves at about eight duct-heights (the second point from the right) which may indicate either the approach of the end of the duct (at ten duct heights) or the beginning of secondary motions.

Because of the clearly different initial conditions, it is not reasonable to compare the curves quantitatively before roughly $\theta = 0{\cdot}4$; because of the anamolous behaviour, it is not reasonable to compare them after $\theta \sim 0{\cdot}6$. Within this range, the agreement appears to be within 20 % for the Reynolds stress, and considerably better for the intensities.

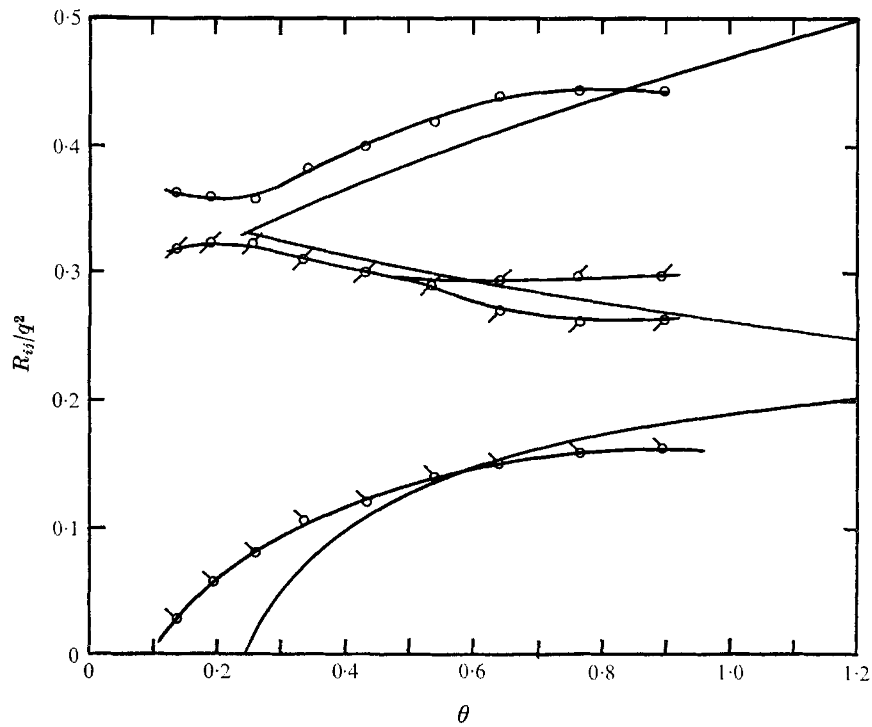


FIGURE 1. Plot of results of Rose *versus* $\theta$. Curves without experimental points are computations of $R_{ij}/q^2$ from the model described in the text with $R_{11}/q^2 = R_{22}/q^2 = R_{33}/q^2 = \frac{1}{3}$, $R_{12}/q^2 = 0$ at $\theta = 0{\cdot}24$. O, $R_{11}/q^2$; ϙ, $R_{22}/q^2$; ơ, $R_{33}/q^2$; ƅ, $R_{12}/q^2$.

## 6. Pure strain

Emboldened by our success in predicting the structure of a homogeneous shear, let us try to predict the evolution of a pure strain. For a two-dimensional strain rate, the equations are

$$\left.\begin{aligned} R'_{11}+(1+\theta)R_{11} &= (1-\beta)q^2/3, \quad U_{1,1} = -U_{2,2} = S, \\ R'_{22}+(1-\theta)R_{22} &= (1-\beta)q^2/3, \quad \theta = S\beta q^2/\epsilon, \\ R'_{33}+R_{33} &= (1-\beta)q^2/3, \quad \epsilon = \epsilon_0 e^{-2\beta\tau}. \end{aligned}\right\} \tag{6.1}$$

These are more difficult to solve than the corresponding ones for a homogeneous shear, even though uncoupled, because of the way in which $\theta$ appears. Since we expect $\theta$ to be monotone, we can replace the independent variable by $\theta$, and change to the dependent variables $K = (R_{22}-R_{11})/(R_{22}+R_{11})$, and $W = 2q^2/3(R_{22}+R_{11})$. The equations become

$$\left.\begin{aligned} dK/d\theta &= [\theta(1-K^2)-(1-\beta)KW]/\theta(\beta+2\theta K/3W), \\ dW/d\theta &= [K\theta(\tfrac{2}{3}-W)+(1-\beta)W(1-W)]/\theta(\beta+2\theta K/3W). \end{aligned}\right\} \quad (6.2)$$

These were solved by the Runge–Kutta method on an IBM 360/67, to give the curves shown in figure 2. The initial conditions were $K = 0$, $W = 1$ at $\theta = 0$, 0·2, 0·4 and 0·6. Solutions were obtained for values of $\beta$ lying between 0·16 and 0·32 by steps of 0·02. The curve shown is for $\beta = 0{\cdot}24$; in the asymptotic region (say for $\theta > 1{\cdot}8$) the variation in $\beta$ affected the third significant figure only.

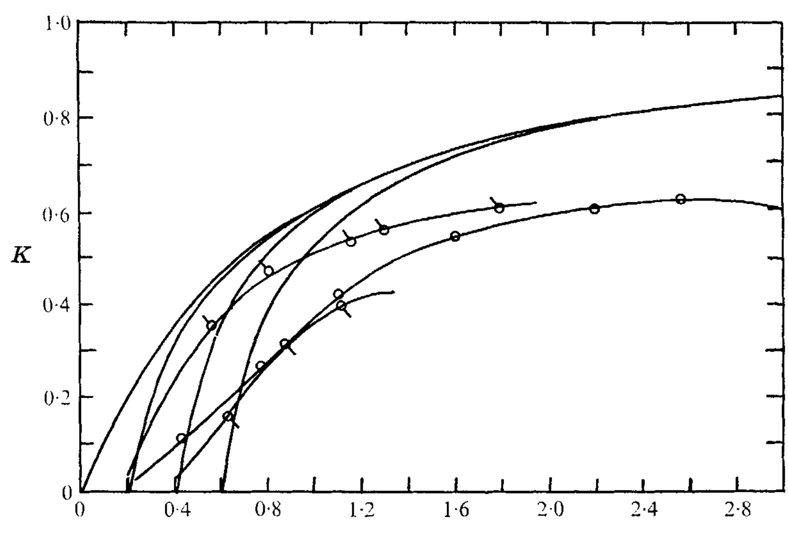


FIGURE 2. Plot of results of Townsend, Tucker & Reynolds and Maréchal *versus* $\theta$. Curves without experimental points are computations of $K$ from the model described in the text, with $K = 0$ initially at $\theta = 0$, 0·2, 0·4, 0·6. Q, Townsend ($\frac{1}{2}$ inch); ○, Maréchal; ȯ, Tucker & Reynolds.

## 7. Discussion and Reynolds number effects

Also shown on figure 2 are the curves obtained from the experiments of Townsend (1954), Maréchal (1967) and Tucker & Reynolds (1968). In each case the independent variable was obtained by integrating equation (3.12) using the measured values of $q^2$.

The three experimental curves appear to be approaching a single asymptotic curve when plotted in this way, as predicted by (2.1), if one ignores the drop-off in each case associated with reduction of strain rate caused by approach of the end of the duct. The general shape of the theoretical curve is very similar to the experimental ones, so that we may conclude that the reasoning presented here is qualitatively correct: that the Reynolds stress is similar in a turbulent time scale, and that that scale continually evolves, so that the Reynolds stress does also.

It is disappointing that the theoretical curve does not agree quantitatively with the experiments. We have made two basic assumptions: our assumption relating the pressure-velocity correlation to the Reynolds stress, and the assumption of high Reynolds number relative to the structure of the dissipating region. The good agreement with Rose's experiment leads one to believe that the assumption of alignment of principal axes is satisfactory. The anisotropy in the pure strain experiment is not significantly greater than that in Rose's experiment, so that the assumption of linearity does not appear to be suspect. The insensitivity to the value of $\beta$ makes it appear unlikely that our choice of value for $\beta$, or even the assumption of its constancy, is at fault. Hence, we can tentatively eliminate the pressure-velocity correlation assumption as a possible cause. In fact, as we have seen from our prediction of the return to isotropy, if anything our assumption predicts too fast a return to isotropy, whereas to reduce the predicted value of $K$ would require an even greater rate.

The other assumption has to do with the influence of the strain rate on the structure of the dissipative region, as a function of Reynolds number. This influence is proportional to $R_{L\epsilon}^{-\frac{1}{2}}$ according to (3.6). For the various experiments the value of $R_{L\epsilon}^{\frac{1}{2}}$ is as follows:

| | |
|---|---|
| Townsend | 6·6 |
| Tucker & Reynolds | 28 |
| Rose | 40 |
| Maréchal | 60 |

A crude estimate indicates that one would expect a reduction in the value of $K$ in each case (from the infinite Reynolds number limit) by roughly 20 %, 4 %, 3 % and 2 %. Our excellent agreement with Rose's experiment is thus explained, but the lack of agreement with the pure strain experiments is not. If one examines the growth rate of $\theta$, which was measured by Townsend (1954), Rose (1966) and Maréchal (1967), we estimate that the additional growth rate due to production is reduced in each case by (respectively) 127 %, 30 %, 21 % and 14 %. Qualitative comparison of the predicted and measured values of $\theta$ indicates that these estimates are reasonable;† from Townsend's measurements, one can see that the growth rate of $\theta$ actually begins to fall below the strain-free decay curve, in agreement with our estimate. Although true values of $\theta$ apparently may be substantially smaller than our estimate, Reynolds number effects will evidently not reduce the value of $K$ sufficiently in the asymptotic region to explain the disagreement between Maréchal's experiment in particular, and our prediction.‡

† Quantitative comparison is difficult. For example, $\theta$ estimated from measured microscales in Rose's experiment, and $\theta$ estimated from $q^2$ using (3.12) differ by nearly a factor of four; those estimated from $q^2$ agree with the value calculated at an equilibrium point.

‡ These Reynolds number effects do provide the possibility, as suggested by Townsend (1954), of attaining an equilibrium, but the values of $\theta$ and $K$ would depend on Reynolds number, $\theta \sim R_{L\epsilon}^{\frac{1}{2}}$.

One final possibility suggests itself; that the experiments, and not the calculation, are in error. For this to be the case in three different experiments would require a systematic failure of the experiments to be what they appear, for some inherent reason. One possible reason is the presence of secondary motion: if a large secondary motion existed, flowing from the contracting walls to the expanding walls, the core of the flow could well experience a strain rate consistently less than that dictated by the wall geometry, without the mean velocity on the axis being affected. None of these experiments reported measurements which would preclude such a secondary motion.

## 8. Conclusions

We can conclude that turbulence under homogeneous deformation behaves like a classical non-linear viscoelastic fluid, but having a monotone increasing time scale, so that no equilibrium structure can be achieved. A simple model predicts behaviour well quantitatively in homogeneous shear, and qualitatively in pure strain. The failure to predict quantitatively in pure strain may be due to the presence of undetected secondary motions.

What can we say about real flows, which are in equilibrium, but which are not homogeneous? In a real flow, the dissipation is maintained at a constant value by spatial transport, so that the time scale does not evolve if the energy is constant. We cannot apply (2.1) directly to such a flow, since the turbulence scales are of the order of the scales of the mean motion, and hence the effects of inhomogeneity are strong. If, however, the big eddies are removed as suggested by Townsend (1956) (see also Lumley 1965), then the remaining turbulence may be of a scale small enough to permit the application of (2.1). We would then conclude, with Townsend, that the structure should be universal, since the time scale is constant and bears a universal ratio to the time scale of the mean motion. (We recall that in a flow truly in equilibrium, we expect all time scales to be proportional with universal constants.)

The qualified success of our model suggests that it may have broader applicability. In particular, it may be useful to predict the small changes in Reynolds stress caused by the imposition of secondary motions (big eddies) on the basic mean motion. As such, it has the desirable property of displaying viscoelastic behaviour, as Townsend (1966) has concluded is necessary.

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## Appendix A. The possibility of a constitutive relation

### A1. *The influence of boundary and initial conditions*

Modern rational mechanics adopts the following philosophical position toward constitutive relations: if they must be determined phenomenologically, they must at least satisfy certain general principles of symmetry, invariance and so forth. In this way a general structure is obtained, to which the phenomenological law must conform. Having such a general structure, it is often possible to predict the stress in one flow from another of the same class, but with a different geometry. By comparison of the general structures implied by requiring satisfaction of different principles, it is often possible to identify causes in experimental results. These remarks all appear to be relevant to the turbulence problem.

We will attempt to apply the reasoning of rational mechanics to the problem of relating the Reynolds stress to the mean velocity field in turbulent motion, the analogue of the classical problem of the constitutive relation. It should be borne in mind, however, that exactly the same analysis can be applied to the problem of relating any other statistical function of the fluctuating velocity field to the mean motion, such as the quantity $\overline{u_{i,j}u_{k,j}}$; the only difference is in the selection of the relevant time and length scales (see § A 3).

The reasoning in this section is an outgrowth of Lumley (1967*b*), and supercedes that paper, primarily due to the material in § A 2.

Usually, when one determines a constitutive relation phenomenologically, one does not have access to the dynamical equations which provide a detailed description of the microstructure, the behaviour of which is reflected in the constitutive relation. In turbulence we are fortunate in having access to these equations, although we cannot solve them. Thus, in any flow, one may write

$$\dot{u}_i + u_{i,j}\overline{U}_j + \overline{U}_{i,j}u_j + u_{i,j}u_j - \overline{u_{i,j}u_j} = -(1/\rho)P_{,i} + \nu u_{i,jj}, \quad u_{i,i} = 0 \qquad \text{(A 1)}$$

which, for specified $R_{ij} = \overline{u_i u_j}$, $\overline{\mathbf{U}}(\mathbf{x}, t)$ and boundary and initial conditions may, in principle, be solved for $\mathbf{u}(\mathbf{x}, t)$. We may expect something like

$$u_i(\mathbf{x}, t) = \mathscr{F}_i\{\overline{\mathbf{U}}(\mathbf{x}+\boldsymbol{\xi}, t')\mathsf{R}, (\mathbf{x}+\boldsymbol{\xi}, t')\}, \quad |\boldsymbol{\xi}| \geqslant 0, \quad t' \leqslant t, \qquad \text{(A 2)}$$

where $\mathscr{F}$ is a functional depending on the values of $\overline{\mathbf{U}}$ and $\mathsf{R}$ everywhere and at all earlier times, and on the boundary and initial conditions.

One may imagine forming $\mathsf{R}$, which would give one an implicit equation for $\mathsf{R}$; one could then solve this by iteration. Having faith that the process would converge, this would lead to

$$R_{ij}(\mathbf{x}, t) = \mathscr{G}_{ij}\{\overline{\mathbf{U}}(\mathbf{x}+\boldsymbol{\xi}, t')\}, \quad |\boldsymbol{\xi}| \geqslant 0, \quad t' \leqslant t, \qquad \text{(A 3)}$$

dependent also on boundary and initial conditions.

The question of boundary and initial conditions is rather serious, since we cannot hope for anything like a constitutive relation so long as there is dependence on the details of these conditions. From experience with turbulent flows, however, it is equally clear that we cannot avoid some dependence on these conditions: thus, the presence of a boundary determines the shear velocity, a scaling parameter throughout the boundary layer. We must allow the boundary and initial conditions to set the levels of some of the scales in the flow. This is similar to St Venant's principle in solid mechanics: sufficiently far from a boundary only integral properties of the detailed boundary conditions are important. Although it has not been proved in turbulence, we will invoke such a principle. We will assume that sufficiently far from the boundary or after initiation, we may determine the structure of $R_{ij}$ completely by giving $\mathbf{U}(\mathbf{x}, t)$ for all $\mathbf{x}$ and all earlier $t$, the initial and boundary conditions serving at most to set the levels of (scalar) time and length scales. It is evident that close to the boundary this is not possible, since for example, the vanishing there (and outside) of **U** does not entail the vanishing of **u**.

Let a length- and time-scale characteristic of the turbulence be given by $L$, $t_p$. Let $y$ be the distance to the nearest boundary, and $t_d$ a time characteristic of the development of the flow. Then, in order to consider the possibility of a constitutive relation, we require that $t_d \gg t_p$ and $L \ll y$. The second requirement does not arise in boundary-free shear flows, such as jets, wakes and shear layers—here one would in principle apply to the solution of (A 1) not a boundary condition but some restriction on behaviour at infinity, such as requiring $|\mathbf{u}| \to 0$ as $|\mathbf{x}| \to \infty$ if $|\overline{U}_{ij}| \to 0$. Thus, one can imagine in a boundary-free shear flow that specification of **U** everywhere, for $t_d \gg t_p$, could, through a universal functional, specify $R$.

Following Townsend (1956) but making use of the results of the body of the paper, we can define

$$\left.\begin{aligned} t_d &= (l_0/\overline{U}_m)(dl_0/dx)^{-1}, \\ t_p &= 0{\cdot}242 \int_{-\infty}^{+\infty} q^2 dy \bigg/ \int_{-\infty}^{+\infty} (-2\overline{uv}\partial\overline{U}/dy)\,dy, \end{aligned}\right\} \quad \text{(A 4)}$$

where $l_0$ is the lateral scale of the flow and $\overline{U}_m$ is the average mean velocity over the flow section. Then, using the values given by Townsend, modified to conform to appendix A, we have for the ratio $t_p/t_d$ the values 0·058 for the two-dimensional wake and circular jet, and 0·11 for the shear layer. Thus, all these flows appear to meet our requirements. Measurements are difficult to discover for other flows but calculations on a similarity basis indicate that a plane jet might be expected to have a value of $t_p/t_d \sim 0{\cdot}12$, an axisymmetric wake about 0·041, a two-dimensional self-propelled wake 0·03 and an axisymmetric self-propelled wake 0·024.

We can conclude that in the classical boundary-free parallel flows, at least, it is possible to have a relation such as (A 3), in which the functional is universal. In flows near boundaries we never observe $L \ll y$: can anything be done about these flows and flows which are developing too rapidly?

This, of course, immediately calls to mind Townsend's (1956) big eddy concept. Perhaps if one were to decompose the flow into large-scale motions, to be treated deterministically, and small-scale ones, to be treated statistically, one could apply a constitutive relation concept to the small-scale motion. If the decomposition is carried out on a rational basis (Lumley 1967*a*), it is possible to show (Lumley 1970) that something like 20–50 % of the energy will be contained in the first mode, or 'large eddy'. Since the remaining small-scale motion will have the same dissipation, we may expect that $t_p/t_d$ will be reduced to 0·8–0·5 of the value for the flow as a whole. This is unlikely to be sufficient to help a flow in trouble due to too-rapid development. As far as length scales are concerned, such a decomposition seems unlikely to produce a reduction by a factor significantly greater than that for the time scales. Thus we may have to exclude flows near boundaries also. Scales in regions well removed from the boundary, in flows with boundaries, are often small enough to meet our requirements (at least marginally) so that one would expect to be able to describe the Reynolds stress in the outer portion of a boundary layer, for example, by a universal functional dependent only on the mean velocity distribution and the shear velocity, but this would be inadequate in the wall region. This is no more than a recognition that the outer portion of the boundary layer is wake-like (Coles 1956). Thus, it seems unlikely that any sort of constitutive relation could apply to rapidly developing flows or flows near boundaries, and we will exclude such flows from consideration in what follows.

### A 2. *Determinism and material indifference*

Classically, constitutive relations for materials are required to satisfy two principles (Truesdell 1961): The principle of determinism, consisting of two basic restrictions—that the stress is determined by the past only, and that it is determined by the motion in an arbitrarily small neighbourhood of the material point in question. The second principle is that of material indifference: that two motions differing only by arbitrary time-dependent rigid motions must produce the same stress, taking proper account of the transformation of coordinates.

With regard to the principle of determinism, we must surely keep the half that says that the present is determined by the past. Just as surely, however, we must discard the half that restricts the causes of stress at a point to an arbitrarily small neighbourhood of that point. All experimental evidence indicates that Reynolds stress (or any other mean quantity in a turbulent flow) is determined by a neighbourhood with a radius of the order of an integral scale. The loss of this half of the principle to determinism does not appear to be serious, however.

The principle of material indifference raises more serious questions. Truesdell (1961) illustrates the principle by pointing out that we expect Hooke's law governing the elongation of a spring in response to a force to be unchanged by the spring being placed on a rotating table. There appears to be a great deal of confusion regarding this principle in the literature and it is sometimes stated as though it were a universal truth, which it clearly is not. Consider, for example, a classical hard-sphere gas in a steadily rotating frame rotating with angular velocity $\mathbf{\Omega}$ about a fixed axis. Consider orthogonal velocities $u_1$ and $u_2$ in a plane perpendicular to the axis of rotation. If a mean free time is given by $\lambda/c$ (where $\lambda$ is the mean free path, and $c$ the r.m.s. velocity) then the Coriolis acceleration will induce velocities $2\mathbf{\Omega}\times\mathbf{u}\lambda/c$, so that to first order the velocity is $\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}\lambda/c$. If the original velocity field (without rotation) $\mathbf{u}$ was uncorrelated (we are in principal axes of the stress tensor in the absence of rotation), then the off-diagonal term induced by the rotation is $\overline{u_1u_2} = (2\Omega\lambda/c)(\overline{u_1^2}-\overline{u_2^2})$. The intensity difference is $\overline{u_1^2-u_2^2} = -\nu S = -\lambda c S$, where $\nu$ is the kinematic viscosity and $S$ the strain rate producing the intensity difference. Finally transforming to new principal axes, the (principal) intensity difference is $-\lambda c S[1+(2\Omega\lambda/c)^2]$, so that the value of the viscosity becomes $\nu[1+(2\Omega\lambda/c)^2]$. Thus the value of the viscosity depends on the angular velocity, at variance with the principle of material indifference. Truesdell suggests that in this principle we are requiring of the constitutive relation a type of invariance which the equations of motion do not obey; in fact, the equations for the micro-dynamics do not, properly speaking, satisfy this kind of invariance either, but the disparity in the time scales involved is usually so great that it makes no difference. Thus, in our example, one must have rotation rates of the order of $c/\lambda$ before an appreciable effect is felt. For air at standard temperature and pressure, this give periods of rotation of the order of 10 picoseconds, or rotation rates of $6\times 10^{12}$ rev/min for an appreciable effect. We may conclude that the principle of material indifference is quite justified for ordinary materials under ordinary conditions.

It is equally obvious, however, that for a turbulent flow, the principle of material indifference is *not* satisfied. Let us imagine the same experiment, say a steady homogeneous pure plane strain, in a steadily rotating framework (angular velocity $(0, 0, \Omega)$) the flow taking place in planes perpendicular to the axis of rotation. The equations for the Reynolds stress $R_{jk}$ become

$$\overline{U}_{i,j}R_{jk}+\overline{U}_{k,j}R_{ji}+2\Omega(\epsilon_{i3l}R_{kl}+\epsilon_{k3l}R_{il}) = -1/\rho(\overline{u_k p_{,i}}+\overline{u_i p_{,k}})-2\nu\overline{u_{i,j}u_{k,j}}. \quad \text{(A 5)}$$

Applying the approximations of the body of the paper, it is a straight-forward calculation to show that

$$\left.\begin{aligned} &R_{13} = R_{23} = 0, \quad R_{33} = q^2(1-\beta)/3, \quad R_{12} = \beta q^2\Omega/S,\\ &R_{11} = (q^2(1-\beta)/3)[1+6(\beta^2/(1-\beta))(\Omega^2q^2/\epsilon S)]/(1+\beta Sq^2/\epsilon),\\ &R_{22} = (q^2(1-\beta)/3)[1-6(\beta^2/(1-\beta))(\Omega^2q^2/\epsilon S)]/(1-\beta Sq^2/\epsilon),\end{aligned}\right\} \quad \text{(A 6)}$$

with $S(R_{11}-R_{22}) = -\epsilon$ and

$$3/(1-\beta) = 1+2[1-6(\beta^3/(1-\beta))(\Omega/S)^2(Sq^2/\epsilon)^2]/[1-\beta^2(Sq^2/\epsilon)^2]. \quad \text{(A 7)}$$

Here, $\beta$ is a real number, $0 \leqslant \beta \leqslant 1$. An examination of (A 7) in the light of this restriction indicates that this approximation certainly does not work for all combinations of $S$, $q^2$, $\epsilon$ and $\Omega$, being restricted in general to values of $(Sq^2/\epsilon)^2 > 1$ (in steady-state flow in an inertial framework, this is observed experimentally to be roughly 10 however (Rose 1966), so this is no restriction) and $(\Omega/S)^2 \leqslant 13/36$. Using $(Sq^2/\epsilon)^2 = 10$, and taking first-order terms in $(\Omega/S)^2$ gives $\beta \sim 0{\cdot}14-2{\cdot}27(\Omega/S)^2+O((\Omega/S)^4)$. Substitution of this value into equation (A 6) indicates that the effect of the rotation depends on a term of the order of $\frac{1}{2}(\Omega/S)^2$. Thus, although our approximation has a somewhat restricted range of applicability (in shear flows, $(\Omega/S)^2 \sim 1$), this range is sufficient to indicate that the effect of rigid rotation on the Reynolds stress is serious in most cases of interest. Consequently, we must discard the principle of material indifference.

What do we have left? Let us restate (A 3) in a form which will allow us to make comparisons with the classical theory. If we define the position of a material point due to displacement by the mean motion by

$$X_i(\mathbf{x}, t/t') = x_i+\int_t^{t'} \overline{U}_i(\mathbf{X}(\mathbf{x}, t/t''), t'')\,dt'', \quad \text{(A 8)}$$

then it is clearly equivalent to write

$$R_{ij}(\mathbf{x}, t) = \mathscr{H}_{ij}\{\mathbf{X}(\mathbf{x}+\boldsymbol{\xi}, t/t')-\mathbf{X}(\mathbf{x}, t/t')\}, \quad |\boldsymbol{\xi}| \geqslant 0, \quad t' \leqslant t, \quad \text{(A 9)}$$

i.e. the stress at a material point is dependent on the history of the displacement of other material points relative to the point in question. Finally, it is equivalent to write

$$R_{ij}(\mathbf{x},t) = \mathscr{I}_{ij}\{X_{p,q}(\mathbf{x}+\boldsymbol{\xi}, t/t')\}, \quad |\boldsymbol{\xi}| \geqslant 0, \quad t' \leqslant t, \tag{A 10}$$

where $X_{p,q} = \partial X_p/\partial x_q$. This is now in the form of a classical constitutive relation, before the restrictions on form due to the principle of material indifference are applied (and overlooking the fact that the functional extends over space, in violation of the principle of determinism).

We can now apply the so-called polar decomposition theorem, which states that the deformation tensor $\mathsf{F} = \partial \mathbf{X}/\partial \mathbf{x}$ can be decomposed uniquely into two tensors

$$\mathsf{F} = \mathsf{RU}, \tag{A 11}$$

where $\mathsf{R}$ is an orthgonal tensor, representing a length-preserving pure rotation, and $\mathsf{U}$ is a symmetric tensor with non-negative eigenvalues, representing a pure strain. We can write

$$\mathsf{F}^T\mathsf{F} = \mathsf{UR}^{-1}\mathsf{RU} = \mathsf{UU} = \mathsf{U}^2, \tag{A 12}$$

which is called $\mathsf{C} = \mathsf{U}^2$, the right Cauchy–Green tensor (to distinguish it from the left Cauchy–Green tensor, formed from the decomposition $\mathsf{F} = \mathsf{VR}$, leading to $\mathsf{FF}^T = \mathsf{VRR}^{-1}\mathsf{V} = \mathsf{VV}$; i.e. rotate after deforming, rather than before), and is simply the metric tensor of the state at $t'$ relative to that at $t$. The components of $\mathsf{C}$ are given by

$$C_{ik} = X_{j,i}X_{j,k}. \tag{A 13}$$

Since the principal axes of $\mathsf{U}$ and those of $\mathsf{C}$ are the same, and the eigenvalues of $\mathsf{C}$ are the squares of those of $\mathsf{U}$ (which are non-negative) it is irrelevant which is used; $\mathsf{C}$ is a rational function of the components of the deformation tensor, while $\mathsf{U}$ is not, so that it is customary to use $\mathsf{C}$. The rotation tensor, $\mathsf{R}$, always has only one real eigenvector of eigenvalue $= +1$: if this is taken as the 3 axis, it may be written as

$$\mathsf{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A 14}$$

for rotation through an angle $\theta$. Hence, this may be replaced by a pseudo vector $\boldsymbol{\Omega}$, of magnitude $\Omega = \sin\theta$, aligned with the axis of rotation, given by

$$\Omega_i = \tfrac{1}{2}\epsilon_{ijk}R_{jk}. \tag{A 15}$$

Knowledge of this pseudo vector will permit reconstruction of $\mathsf{R}$ as follows: Take $\boldsymbol{\Omega}/\Omega$ as an eigenvector, and generate any two other orthogonal to it and each other, say $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$: then $\mathsf{R}$ is given by

$$R_{ij} = (1-\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})^{\frac{1}{2}}(X_i^{(1)}X_j^{(1)} + X_i^{(2)}X_j^{(2)}) + \Omega_i\Omega_j/\Omega^2 + \epsilon_{ijk}\Omega_k. \tag{A 16}$$

Hence, we may write

$$R_{ij}(\mathbf{x},t) = \mathscr{I}_{ij}\{\mathsf{C}(\mathbf{x}+\boldsymbol{\xi}, t/t'), \boldsymbol{\Omega}(\mathbf{x}+\boldsymbol{\xi}, t/t')\}, \quad |\boldsymbol{\xi}| \geqslant 0, \quad t' \leqslant t \tag{A 17}$$

in place of (A 10). In the classical theory of constitutive relations, the principle of material indifference would now be used to eliminate dependence on $\boldsymbol{\Omega}$. It must be noted that, in order for our earlier discussions to make sense, position $\mathbf{x}$ in (A 17) must be defined relative to an inertial framework.

### A 3. *Time and length scales*

Both $\mathbf{C}$ and $\mathbf{\Omega}$ in expression (A 17) are dimensionless. Only one parameter is provided by the equations of motion, the viscosity $\nu$. Hence as discussed in §A 1, the boundary and initial conditions must provide a length scale and a time scale with which to non-dimensionalize the length and time which appear in (A 17); together with $\nu$ a Reynolds number can be formed from these, on which we also expect (A 17) to be dependent. Finally, these will form a dimensional multiplicative factor. There may, of course, be more than one length and time scale present in the flow; the ratios of the additional ones to the primary ones will then appear as parameters also.

In many flows, particularly those of interest to us here, it is difficult to identify from the boundary and initial conditions the relevant length and time scales. It is completely equivalent to define a length and time from the statistics of the turbulence—these scales will bear a unique relation to the scales determined by the boundary and initial conditions. Thus, we may pick in a turbulent flow the quantities $q^2$ and $\epsilon$ as fundamental, forming a length from these, $L_\epsilon = q^3/3^{\frac{3}{2}}\epsilon$, and a time $\beta q^2/2\epsilon$. If we now define a new time scale $\tau$, and a new length scale $\zeta$ by

$$d\tau/dt = 2\epsilon/\beta q^2, \quad d\zeta/dx = 1/L_\epsilon, \tag{A 18}$$

then (A 17) can be written as

$$R_{ij}(\mathbf{x},t) = \mathscr{R}_{ij}(\boldsymbol{\zeta},\tau) = q^2\mathscr{K}_{ij}\{\mathbf{C}(\boldsymbol{\zeta}+\boldsymbol{\zeta}',\tau/\tau'),\mathbf{\Omega}(\boldsymbol{\zeta}+\boldsymbol{\zeta}',\tau/\tau'),R_{L\epsilon}\},$$
$$|\boldsymbol{\zeta}'| \geqslant 0, \quad \tau' \leqslant \tau, \tag{A 19}$$

which is also assumed to be a function of time- and length-scale ratios, if others are present.

Of course, as long as (A 19) remains a function of the Reynolds number and/or other length- and time-scale ratios, the functional will not be universal; it will be universal only when a single length and time scale serve to describe the turbulence. When this is true, we say the turbulence is self-preserving, since its evolution is also governed by the same scales, and the expression (A 19) is also self-preserving.

It should be pointed out that this analysis is equally applicable to other statistical quantities than $R_{ij}$; for these, of course, the appropriate length and time scales must be chosen differently.

### A 4. *Homogeneous shear: a 'simple fluid'*

As is done in continuum mechanics, let us consider some simple flows. Specifically, let us consider steady homogeneous shear, and steady homogeneous pure strain. The assumption of steadiness will provide considerable simplification, but will not eliminate the effect of unsteadiness, since (A 17) or (A 19) is expressed in Lagrangian co-ordinates—only trivial flows are steady in Lagrangian co-ordinates.

Consider first the steady homogeneous shear. We have

$$\bar{\mathbf{U}} = \{U'x_2, 0, 0\}, \tag{A 20}$$

which gives

$$\mathsf{C} = \begin{pmatrix} 1 & U'(t'-t) & 0 \\ U'(t'-t) & 1+U'^2(t'-t)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} 0 \\ 0 \\ f(U'(t'-t)) \end{pmatrix}, \tag{A 21}$$

where $f$ is a rather complicated function of $U'(t'-t)$. Thus, $\mathsf{C}$ is a linear combination of three constant tensors

$$\begin{aligned} \mathsf{C} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + U'(t'-t)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + U'^2(t'-t)^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathsf{I} + U'(t'-t)\mathsf{A}^{(1)} + U'^2(t'-t)^2\mathsf{A}^{(2)} \end{aligned} \tag{A 22}$$

and $\boldsymbol{\Omega}$ is proportional to a constant vector

$$\boldsymbol{\Omega} = f(U'(t'-t))\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = f(U'(t'-t))\boldsymbol{\lambda}. \tag{A 23}$$

Thus we can write $\quad \mathscr{F}_{ij}\{\mathsf{C}, \boldsymbol{\Omega}) = F_{ij}(\mathsf{A}^{(1)}, \mathsf{A}^{(2)}, \boldsymbol{\lambda}) = R_{ij},$ (A 24)

a function of two symmetric tensors and a vector.

From the analysis of appendix B, the function in (A 24) must have the form

$$R_{ij} = a\delta_{ij} + bA^{(1)}_{ij} + cA^{(1)2}_{ij} + dA^{(2)}_{ij} + eA^{(2)2}_{ij}. \tag{A 25}$$

The $a, \ldots, e$ are functions of the invariants of $\mathsf{A}^{(1)}$, and of $\lambda_i\lambda_i$, $A^{(1)}_{ij}\lambda_i\lambda_j$, $A^{(1)2}_{ij}\lambda_i\lambda_j$, $A^{(2)}_{ij}\lambda_i\lambda_j$, $A^{(2)2}_{ij}\lambda_i\lambda_j$ and of the shear $U'$. In our case, all invariants are numerical, so that $a, \ldots, e$ are functions only of $U'$. Evaluating the various terms in (A 25) we have

$$R_{ij} = \begin{pmatrix} a+c & b & 0 \\ b & a+c+d+e & 0 \\ 0 & 0 & a \end{pmatrix}. \tag{A 26}$$

This is the same behaviour as Coleman & Noll's (1961) 'simple fluid', a very general model of a non-Newtonian fluid; the dependence on rotation does not introduce any greater generality. It is encouraging that we could have arrived at the form (A 26) on the basis of symmetry.

### A 5. *Viscoelasticity*

We can learn something about the viscoelastic character of turbulence by asking what form (A 17) or (A 19) would take if the turbulence had no memory. This is, of course, an artificial situation. In order to imagine a fluid with no memory, we must imagine that somehow we cause the time scale $q^2/2\epsilon$ to become shorter and shorter relative to some time characteristic of the mean motion. Since $q^2/2\epsilon$ is ordinarily determined by the time scale characteristic of the mean motion, this is intrinsically impossible. As a thought experiment, however, it is instructive.

Going back to (A 17), if the fluid has no memory, $\mathsf{R}$ will be a function only of $(\partial\mathsf{C}/\partial t')_{t'=t}$, $(\partial\boldsymbol{\Omega}/\partial t')_{t'=t}$. Carrying out the operations gives

$$(\partial C_{ij}/\partial t')_{t'=t} = U_{i,j} + U_{j,i} = 2S_{ij}; \quad (\partial\Omega_i/\partial t')_{t'=t} = -\tfrac{1}{2}\omega_i, \tag{A 27}$$

where $S_{ij}$ is the strain rate tensor, and $\omega_i$ the vorticity vector. Now, in a homogeneous deformation, in which $S_{ij}$ and $\omega_i$ are constant everywhere, we have

$$R_{ij} = H_{ij}(\mathsf{S}, \boldsymbol{\omega}), \tag{A 28}$$

i.e.—a function of a symmetric tensor and a vector. Referring to appendix B again, we see that this must have the form

$$R_{ij} = a\delta_{ij} + bS_{ij} + cS^2_{ij}, \tag{A 29}$$

where $a, b$ and $c$ are functions of the invariants of $\mathsf{S}$, say $S^2_{ii}$ and $S^3_{ii}$ (since $S_{ii} = 0$), as well as of the invariants $\omega^2$, $S_{ij}\omega_i\omega_j$, $S^2_{ij}\omega_i\omega_j$. This has the form of a Reiner–Rivlin fluid (the dependence on $\boldsymbol{\omega}$ does not introduce any additional complexity).

In a steady simple shear, we have

$$S_{ij} = (\tfrac{1}{2}U)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^2_{ij} = (\tfrac{1}{2}U)^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A 30}$$

so that

$$R_{ij} = \begin{pmatrix} a + c(\tfrac{1}{2}U')^2 & b\tfrac{1}{2}U' & 0 \\ b\tfrac{1}{2}\overline{U}' & a + c(\tfrac{1}{2}U')^2 & 0 \\ 0 & 0 & a \end{pmatrix}. \tag{A 31}$$

Hence, $R_{11} = R_{22}$, contrary to observation.

From a comparison of (A 31) with (A 26), we may conclude that the inequality of $R_{11}$ and $R_{22}$ in real flows is positive evidence of the dependence of Reynolds stress on history, i.e. positive evidence for viscoelastic behaviour. This evidence is thus complementary to that of Moffatt (1965) and Crow (1968), indicating viscoelastic behaviour for small times and small strains, since this is evidence of viscoelasticity when non-linearity is dominant.

### A 6. *Pure homogeneous strain*

Turning now to the case of a steady, pure, homogeneous strain, we have $\overline{U}_{i,j} = S_{ij} = S_{ji} = \text{const.}, \boldsymbol{\Omega} = 0$. It is easy to show that, if $S_{ij}\xi^{(k)}_j = \lambda^{(k)}\xi^{(k)}_i$, the right Cauchy–Green tensor becomes

$$C_{jk} = \sum_{l=1}^{3} e^{-2\lambda^{(l)}(t'-t)}\xi^{(l)}_j\xi^{(l)}_k = e^{-2(t'-t)S_{jk}} \quad \text{(by definition)}. \tag{A 32}$$

Hence we can write

$$\begin{aligned} R_{ij} &= \mathscr{I}_{ij}\{e^{-2(t'-t)\mathsf{S}}, \mathbf{0}\} = F_j(\mathsf{S}) \\ &= a\delta_{ij} + bS_{ij} + cS^2_{ij} \end{aligned} \tag{A 33}$$

(using the results of appendix B) where $a$, $b$ and $c$ are functions of the invariants $S^2_{ii}$ and $S^3_{ii}$ (since $S_{ii} = 0$). This has the form of a Reiner–Rivlin fluid. The form is the same as the one we obtained when we assumed no dependence on history; hence we would not be inclined to suspect effects of viscoelasticity here. If we assume that the fluid behaviour is characterized by a single time constant, however (as in the body of the paper), we must conclude that the degree of anisotropy in response to the strain rate field is dependent on the ratio of the time scale to the time characterizing the strain rate. This is a viscoelastic effect, even though it enters implicitly rather than explicitly.

## Appendix B. Form of a second rank tensor function of two symmetric second rank tensors and a vector

B 1. *Invariant basis for two symmetric second rank tensors and three vectors, under proper orthogonal transformations in three dimensions*

Consider three vectors and two symmetric tensors

$$\boldsymbol{\phi},\quad \boldsymbol{\psi},\quad \boldsymbol{\Omega},\quad \mathsf{A}^{(1)},\quad \mathsf{A}^{(2)}. \tag{B 1}$$

An invariant basis will be formed by the components of all of these reduced to the principal axes of $\mathsf{A}^{(1)}$ say. This would give the following numbers of independent invariants for each quantity:

$$\left.\begin{array}{ll} \mathsf{A}^{(1)}: & 3 \\ \mathsf{A}^{(2)}: & 6 \\ \boldsymbol{\Omega}: & 3 \\ \boldsymbol{\psi}: & 3 \\ \boldsymbol{\phi}: & \underline{3} \\ & 18 \end{array}\right\} \tag{B 2}$$

If $I^{(1)}$, $II^{(1)}$, $III^{(1)}$ are the three principal invariants of $\mathsf{A}^{(1)}$, then the requisite invariants relating to $\mathsf{A}^{(1)}$, may be obtained from the quantities

$$\left.\begin{array}{l} I^{(1)},\quad II^{(1)},\quad III^{(1)},\quad \Omega_i\Omega_i,\quad A^{(1)}_{ij}\Omega_i\Omega_j,\quad A^{(1)^2}_{ij}\Omega_i\Omega_j, \\ \Omega_i\psi_i,\quad A^{(1)}_{ij}\Omega_i\psi_j,\quad A^{(1)^2}_{ij}\Omega_i\psi_j,\quad \phi_i\psi_i,\quad A^{(1)}_{ij}\phi_i\psi_j,\quad A^{(1)^2}_{ij}\phi_i\psi_j. \end{array}\right\} \tag{B 3}$$

These twelve relationships should take care of all invariants of $\mathsf{A}^{(1)}$, $\boldsymbol{\Omega}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$. Closer examination discloses that the first six will permit determination of $\Omega^{(1)^2}$, $\Omega^{(2)^2}$, $\Omega^{(3)^2}$ (the components of $\boldsymbol{\Omega}$ in the principal axes of $\mathsf{A}^{(1)}_i$); the addition of the next three will permit determination of $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$ *to within a reflexion* represented by the sign ambiguity. We need one more relation to resolve this ambiguity:

$$\epsilon_{ijk}\Omega_i\phi_j\psi_k \tag{B 4}$$

resolves it, since this changes sign under reflexion.

We now need six more relations that will permit determination of the six components of $\mathsf{A}^{(2)}$ in the principal axes of $\mathsf{A}^{(1)}$. These are

$$\left.\begin{array}{ll} A^{(2)}_{ij}\phi_i\psi_j, & A^{(2)^2}_{ij}\phi_i\psi_j, \\ A^{(2)}_{ij}\Omega_i\psi_j, & A^{(2)^2}_{ij}\Omega_i\psi_j, \\ A^{(2)}_{ij}\Omega_i\Omega_j, & A^{(2)^2}_{ij}\Omega_i\Omega_j, \end{array}\right\} \tag{B 5}$$

which will permit determination of all six components. Thus, the total list required consists of (B 3), (B 4), and (B 5).

B 2. *Use of the invariant basis to determine the structure of a second rank tensor function*

In considering the form of a second rank tensor function $\tau_{ij}$ of the variables $\mathsf{A}^{(1)}$, $\mathsf{A}^{(2)}$ and $\boldsymbol{\Omega}$, we first form an invariant with two arbitrary vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, $\tau_{ij}\phi_i\psi_j$. This invariant must now be a function of the invariant basis above (B 3), (B 4) and (B 5). However, since $\tau_{ij}\phi_i\psi_j$ is bilinear in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, certain of the invariants in our list may be immediately eliminated, specifically, those in which $\boldsymbol{\psi}$ appears in first degree without $\boldsymbol{\phi}$. We could not have eliminated these if we also had ones in which $\boldsymbol{\phi}$ appeared to first degree without $\boldsymbol{\psi}$, since the product would be bilinear; we have managed to pick a sufficient set of invariants, however, without doing this. Hence $\tau_{ij}\phi_i\psi_j$ will be a function of

$$I^{(1)},\quad II^{(1)},\quad III^{(1)},\quad \Omega_i\Omega_i,\quad A^{(1)}_{ij}\Omega_i\Omega_j,\quad A^{(1)2}_{ij}\Omega_i\Omega_j,\quad A^{(2)}_{ij}\Omega_i\Omega_j,\quad A^{(2)2}_{ij}\Omega_i\Omega_j,$$
$$\phi_i\psi_i,\quad A^{(1)}_{ij}\phi_i\psi_j,\quad A^{(1)2}_{ij}\phi_i\psi_j,\quad A^{(2)}_{ij}\phi_i\psi_j,\quad A^{(2)2}_{ij}\phi_i\psi_j,\quad \epsilon_{ijk}\Omega_i\phi_j\psi_k. \qquad \text{(B 6)}$$

Of these, the ones on the first line do not contain $\boldsymbol{\phi}$ or $\boldsymbol{\psi}$, while those on the second line are bilinear in them. Thus $\tau_{ij}\phi_i\psi_j$ must be a linear function of those on the second line, with coefficients functions of those on the first, Hence.

$$\tau_{ij} = a\delta_{ij} + bA^{(1)}_{ij} + cA^{(1)2}_{ij} + dA^{(2)}_{ij} + eA^{(2)2}_{ij} + f\epsilon_{ijk}\Omega_k, \qquad \text{(B 7)}$$

where $a, \ldots, f$ are functions of the invariants on the first line in (B 6).

If $\tau_{ij}$ is a stress tensor, it must be symmetric; hence $f = 0$.

Thus, referring to appendix A, the *structure* of the stress tensor will be the same whether $\boldsymbol{\Omega}$ is included or not, although the invariant functions will differ.

## REFERENCES


Batchelor, G. K. 1956 *Homogeneous Turbulence*. Cambridge University Press.

Coleman, B. D. & Noll, W. 1961 Recent results in the continuum theory of viscoelastic fluids. *Ann. N.Y. Acad. Sci.* **89**, 672–714.

Coles, D. 1956 Law of the wake and law of the wall. *J. Fluid Mech.* **1**, 191.

Crow, S. C. 1968 Viscoelastic properties of fine grained incompressible turbulence. *J. Fluid Mech.* **33**, 1–20.

Grant, H. L. 1958 The large eddies of turbulent motion. *J. Fluid Mech.* **4**, 149–191.

Liepmann, H. W. 1961 Free turbulent flows. *Mécanique de la Turbulence.* Paris: C.N.R.S.

Lumley, J. L. 1967*a* The structure of inhomogeneous turbulent flows. *Atmospheric Turbulence and Radio Wave Propagation.* Moscow: Nauka.

Lumley, J. L. 1967*b* On a rational approach to relations between motions of different scales in turbulent flows. *Phys. Fluids,* **10**, 1405–1408.

Lumley, J. L. 1970 *Stochastic Tools in Turbulence.* New York: Academic.

Maréchal, J. 1967 Anisotropie d'une Turbulence de grille déformée par un champ de vitesse moyenne homogène. *C.R. Acad. Sci. Paris*, A **265**, 478–480.

Moffatt, H. K. 1965 *The Interaction of Turbulence with Rapid Uniform Shear. SUDAER Rep.* no. 242, Stanford University.

Rivlin, R. S. 1957 The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. *Quart. Appl. Math.* **15**, 212.

Rose, W. G. 1966 Results of an attempt to generate a homogeneous turbulent shear flow. *J. Fluid Mech.* **25**, 97–120.

Rotta, J. 1951 Statistische theorie nichthomogener turbulenz. *Z. für Physik,* **129**, 547–572.

Townsend, A. A. 1954 The uniform distortion of homogeneous turbulence. *Quart. J. Mech. Appl. Math.* **7**, 104–127.

Townsend, A. A. 1956 *Structure of Turbulent Shear Flows.* Cambridge University Press.

Townsend, A. A. 1966 The mechanism of entrainment in free turbulent flows. *J. Fluid Mech.* **26**, 689.

Truesdell, C. 1961 The principles of continuum mechanics. *Colloquium Lectures in Pure and Applied Science no.* 5, Socony Mobil Oil Co., Inc.

Tucker, H. J. & Reynolds, A. J. 1968 The distortion of turbulence by irrational plane strain. *J. Fluid Mech.* **32**, 657–673.